UPPER AND LOWER BOUNDS FOR THE ITERATES OF ORDER-PRESERVING HOMOGENEOUS MAPS ON CONES

PHILIP CHODROW, COLE FRANKS, AND BRIAN LINS*

Abstract. We define upper bound and lower bounds for order-preserving homogeneous of degree one maps on a proper closed cone in $\mathbb{R}^n$ in terms of the cone spectral radius. We also define weak upper and lower bounds for these maps. For a proper closed cone $C \subset \mathbb{R}^n$, we prove that any order-preserving homogeneous of degree one map $f : \operatorname{int} C \to \operatorname{int} C$ has a lower bound. If C is polyhedral, we prove that the map f has a weak upper bound. We give examples of weak upper bounds for certain order-preserving homogeneous of degree one maps defined on the interior of $\mathbb{R}^n_+$.

1. Introduction

A *closed cone* $C \subset \mathbb{R}^n$ is a closed convex set such that (i) $C \cap (-C) = \{0\}$ and (ii) $\lambda C = C$ for all $\lambda \geq 0$. If C has nonempty interior, we say that C is a *proper closed cone*. For any proper closed cone, the *dual cone* $C^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0 \, \forall y \in C\}$ is a proper closed cone.

Any closed cone C defines a partial ordering on $\mathbb{R}^n$ by $x \leq_C y$ if and only if $y - x \in C$. When the cone C is understood, we will write $\leq$ instead of $\leq_C$. Let $D \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. A map $f : D \to \mathbb{R}^n$ is said to be *order-preserving* if and only if $f(x) \leq f(y)$ whenever $x \leq y$. It is called *order-reversing* if $f(x) \geq f(y)$ whenever $x \leq y$. We say that $f : D \to \mathbb{R}^n$ is homogeneous of degree α if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and $x \in D$. Order-preserving homogeneous of degree one maps from a cone into itself have been extensively studied (see e.g., [6]). They are a natural extension of the nonnegative matrices, and there are many examples of such maps in applications [9]. Many important properties of nonnegative matrices generalize to order-preserving homogeneous of degree one maps on cones.

Let C be a proper closed cone and let $f : C \to C$ be a continuous order-preserving homogeneous of degree one map. We define the *cone spectral radius* of f to be

$$\rho_C = \rho_C(f) = \lim_{k\to\infty} ||f^k(x)||^{1/k}, \tag{1}$$

for any $x \in \operatorname{int} C$. The value of ρ_C is independent of x [6, Proposition 5.3.6]. Once again, when the cone C is understood, we will write ρ instead of ρ_C. The well known Krein-Rutman theorem [6, Corollary 5.4.2] asserts that any continuous order-preserving homogeneous of degree one map $f : C \to C$ has an eigenvector $x \in C$ such that $f(x) = \rho x$. Note that any such eigenvector will serve as a lower bound on the iterates of f in the following sense. If $x \leq y$, then $f^k(x) \leq f^k(y)$ for all $k \in \mathbb{N}$ because f is order-preserving. Thus $f^k(y) \geq \rho^k x$ for all k.

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*Corresponding author.

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In applications, we often cannot say that the map is defined continuously on the entire closed cone. There are several examples of important order-preserving homogeneous of degree one maps that are defined only on the interior of the cone. Suppose now that $f : \operatorname{int} C \to \operatorname{int} C$ is order-preserving homogeneous of degree one map. The spectral radius of f given in (1) is still well defined [7, Theorem 2.2]. Since $f(\operatorname{int} C) \subseteq \operatorname{int} C$, it follows that $\rho(f) > 0$. This is because there exists $\alpha > 0$ such that $f(x) \geq \alpha x$, so $f^k(x) \geq \alpha^k x$. Since C is a closed cone in a finite dimensional space, it is normal [6, Lemma 1.2.5], so there exists a $c > 0$ such that $y \geq x$ implies $||y|| \geq c||x||$. In particular $||f^k(x)|| \geq c||\alpha^k x||$, so $\rho(f) \geq \alpha > 0$.

We say that $y \in C \backslash \{0\}$ is a *lower bound* for f if for all $x \in \operatorname{int} C$, $x \geq y$ implies that $f(x) \geq \rho y$. Note that any eigenvector of f is a lower bound. Unfortunately, the Krein-Rutman theorem does not apply in general if f is only defined on the interior of the cone. Without an eigenvector corresponding to the spectral radius, it is not clear that a lower bound must exist. We address this question in the next section, but for now, let us introduce a weaker notion. We say that $w \in C^*$ is a *weak lower bound* for f if there exists $x \in \operatorname{int} C$ such that $\langle f^k(x), w\rangle \geq \rho^k \langle x, w\rangle$ for all $k \geq 0$.

For $f : \operatorname{int} C \to \operatorname{int} C$ order-preserving and homogeneous of degree one we say that $y \in \operatorname{int} C$ is an *upper bound* if $x \leq y$ implies that $f(x) \leq \rho y$. Unlike lower bounds, we do not allow upper bounds on the boundary of C. After all, if y is contained in the boundary of C, then there is no $x \in \operatorname{int} C$ such that $x \leq y$ so it does not make sense to refer to y as an upper bound. We say that $w \in C^*$ is a *weak upper bound* for f if there exists $x \in \operatorname{int} C$ such that $\langle f^k(x), w\rangle \leq \rho^k \langle x, w\rangle$. A weak upper or lower bound is uniform in the following sense.

Lemma 1. *Let C be a proper closed cone in $\mathbb{R}^n$ and $f : \operatorname{int} C \to \operatorname{int} C$ be order-preserving and homogeneous of degree one. If w is a weak lower (upper) bound for f and $y \in \operatorname{int} C$, then there exists a constant $c > 0$ such that $\langle f^k(y), w\rangle \geq c\rho^k$ (respectively, $\langle f^k(y), w\rangle \leq c\rho^k$).*

Proof. If w is a weak lower bound for f, then there exists $x \in \operatorname{int} C$ such that $\langle f^k(x), w\rangle \geq \rho^k \langle x, w\rangle$. Since both $x, y \in \operatorname{int} C$, there exists $\alpha > 0$ such that $\alpha x \leq y$. Applying the map f^k,

$$\alpha f^k(x) \leq f^k(y)$$
$$\alpha \langle f^k(x), w\rangle \leq \langle f^k(y), w\rangle$$

Therefore

$$\alpha \langle x, w\rangle \leq \alpha \langle f^k(x), w\rangle \leq \langle f^k(y), w\rangle \leq \beta \langle f^k(x), w\rangle$$

Letting $c = \alpha \langle x, w\rangle$ completes the proof if w is a weak lower bound. The proof for weak upper bounds is essentially the same. □

As the following lemma shows, the notion of a weak upper or lower bound is indeed weaker than an upper or lower bound.

Lemma 2. *Let C be a proper closed cone in $\mathbb{R}^n$ and $f : \operatorname{int} C \to \operatorname{int} C$ be order-preserving and homogeneous of degree one. If f has a lower (upper) bound, then there is a weak lower (weak upper) bound on the iterates of f.*

Proof. Suppose that y is a lower bound for f. Then for every $x \in \operatorname{int} C$, there is a maximal $\lambda > 0$ such that $x \geq \lambda y$ and it is clear that λy is also a lower bound. Thus $x - \lambda y \in \partial C$ where ∂C denotes the boundary of C. We may choose $w \in C^* \backslash \{0\}$ such

that $\langle x - \lambda y, w\rangle = 0$ and $\langle x, w\rangle = \langle \lambda y, w\rangle$. Since λy is a lower bound, $f^k(x) \geq \rho^k \lambda y$ for all $k \in \mathbb{N}$. Therefore $\langle f^k(x), w\rangle \geq \rho^k \langle \lambda y, w\rangle = \rho^k \langle x, w\rangle$. The proof for upper bounds is essentially the same. □

We will prove that for any proper closed cone C and any order-preserving homogeneous of degree one map $f : \operatorname{int} C \to \operatorname{int} C$, the map f has a lower bound. For order-preserving homogeneous of degree one map on the standard cone $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in \{1, ..., n\}\}$, we show in section 3 that there is a formal eigenvector that is almost an upper bound for the map. In particular this will establish a weak upper bound for the iterates of any such map on the standard cone. We then extend this result to show that on any polyhedral cone there is always a weak upper bound for the iterates of any order-preserving homogeneous of degree one map defined on the interior.

2. Lower Bounds

Let C be a proper cone in $\mathbb{R}^n$ and let $f : \operatorname{int} C \to \operatorname{int} C$ be order-preserving and homogeneous of degree one. It is known [3, Theorem 2.10] that if C is a polyhedral cone, then f has a continuous extension to C that is order-preserving and homogeneous of degree one. By the Krein-Rutman theorem this extension has an eigenvector $y \in C \backslash \{0\}$ with eigenvalue equal to the cone spectral radius $\rho(f)$. This proves that order-preserving, homogeneous of degree one self-maps of the interior of a closed polyhedral cone must have a lower bound.

When the cone C is not polyhedral, the map f might not extend continuously to the boundary of C. In this case, however, there must still be a lower bound.

Theorem 1. *Let C be any proper closed cone in $\mathbb{R}^n$ and suppose $f : \operatorname{int} C \to \operatorname{int} C$ is order-preserving and homogeneous of degree one with cone spectral radius $\rho(f) = \rho$. There exists $y \in C\backslash\{0\}$ such that for any $x \in \operatorname{int} C$ with $x \geq y$, $f^k(x) \geq \rho^k y$ for all $k \in \mathbb{N}$.*

Proof. Fix $v \in \operatorname{int} C^*$ and $x_0 \in \operatorname{int} C$ and let $f_\epsilon(x) = f(x) + \epsilon\langle x, v\rangle x_0$ for $\epsilon > 0$. By [6, Theorem 5.4.1], each map f_ϵ has an eigenvector $y_\epsilon \in \operatorname{int} C$ with $||y_\epsilon|| = 1$ and $f_\epsilon(y_\epsilon) = \rho_\epsilon y_\epsilon$ where ρ_ϵ is the cone spectral radius of f_ϵ. Furthermore, $\lim_{\epsilon \to 0} \rho_\epsilon = r$ exists. Since $\rho_\epsilon > \rho$ for every ϵ, it follows that $r \geq \rho$. We may choose a sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ such that $\epsilon_i \to 0$ and $y_{\epsilon_i} \to y$ where $y \in C$.

For a fixed $x \in \operatorname{int} C$ with $x \geq y$, and for each y_ϵ, there exists a maximal $\lambda_\epsilon > 0$ such that $x \geq \lambda_\epsilon y_\epsilon$. In particular, $x - \lambda_\epsilon y_\epsilon \in \partial C$, where ∂C denotes the boundary of C. We claim that $\inf_{\epsilon > 0} \lambda_\epsilon > 0$. Since $x \in \operatorname{int} C$, there exists $\delta > 0$ such that $x - \delta z \in \operatorname{int} C$ for all $z \in \mathbb{R}^n$ with $||z|| = 1$. Thus $x - \delta y_\epsilon \in \operatorname{int} C$ for all $\epsilon > 0$. So $x \geq \delta y_\epsilon$ for all ϵ and therefore $\inf \lambda_\epsilon \geq \delta$. By taking a refinement if necessary, we may assume that $\lambda_{\epsilon_i} \to \lambda$ where $\lambda \geq \inf_{\epsilon > 0} \lambda_\epsilon > 0$. Then $x - \lambda_{\epsilon_i} y_{\epsilon_i}$ converges to $x - \lambda y$ and since ∂C is a closed set, $x - \lambda y \in \partial C$. Given that $x \geq y$, $\lambda \geq 1$.

Since $x \geq \lambda_\epsilon y_\epsilon$ for each ϵ, $f_\epsilon^k(x) \geq \lambda_\epsilon f_\epsilon^k(y_\epsilon) = \lambda_\epsilon \rho_\epsilon^k y_\epsilon$ for any $k \in \mathbb{N}$. Taking a limit, we see that $f^k(x) \geq \lambda r^k y \geq \rho^k y$ for all $k \in \mathbb{N}$. □

It has been noted that order-preserving homogeneous of degree one maps on the interior of symmetric cones have weak lower bounds [4, Corollary 21]. The result above implies that both weak lower bounds and lower bounds will exist for order-preserving homogeneous of degree one self-maps of the interior of any proper closed cone.

3. Upper Bounds on the Standard Cone

Even in the standard cone an order-preserving homogeneous of degree one map $f : \operatorname{int}\mathbb{R}^n_+ \to \operatorname{int}\mathbb{R}^n_+$ might not have an upper bound. For example, the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ defines a linear transformation on the standard cone $\mathbb{R}^2_+$ with spectral radius 1, but the iterates of any vector with positive entries under powers of A forms an unbounded sequence. The following theorem shows that if we relax the definition of an upper bound slightly, then we get a kind of upper bound that is not in C but has all of the other properties of an upper bound for the map f.

Theorem 2. *Let $f : \operatorname{int}\mathbb{R}^n_+ \to \operatorname{int}\mathbb{R}^n_+$ be order-preserving, homogeneous of degree one. Let $\rho = \rho(f)$ be the cone spectral radius of f. The map f extends continuously to an order-preserving map on $(0,\infty]^n$ and there exists $z \in (0,\infty]^n$ such that z has at least one finite entry and $f(z) = \tilde{\rho} z$ where $\tilde{\rho} \le \rho$. In particular, if $x \le z$, then $f(x) \le \rho z$.*

We refer to z in the theorem above as a *formal eigenvector* of f. A formal eigenvector satisfies all of the properties of an upper bound for f, except that it is not an element of $\mathbb{R}^n_+$. Before we prove Theorem 2, we need some definitions and a lemma.

In what follows, let $L : [0,\infty]^n$ be the entry-wise reciprocal map:

$$(Lx)_i = \begin{cases} x_i^{-1} & \text{if } x_i \in (0,\infty) \\ x_i = \infty & \text{if } x_i = 0 \\ x_i = 0 & \text{if } x_i = \infty \end{cases} \tag{2}$$

The set $[0,\infty]^n$ has the obvious partial order $x \le y$ if $x_i \le y_i$ for all $i \in \{1,\dots n\}$. Note that L is order-reversing with respect to this partial ordering. Furthermore, $L(\lambda x) = \lambda^{-1} x$ for all $\lambda \in (0,\infty)$ so L is homogeneous of degree -1 on $\operatorname{int}\mathbb{R}^n_+$.

Lemma 3. *Let $f : \operatorname{int}\mathbb{R}^n_+ \to \operatorname{int}\mathbb{R}^n_+$ be an order-preserving homogeneous of degree one map and let L be given by (2). Then $\rho(L \circ f \circ L)^{-1} \le \rho(f)$.*

Proof. For any $x \in \operatorname{int}\mathbb{R}^n_+$ and $k \in \mathbb{N}$, $(L \circ f \circ L)^k(x) = L \circ f^k \circ L(x)$. By the Cauchy-Schwarz inequality, $||Lz||||z|| \ge n$ for all $z \in \operatorname{int}\mathbb{R}^n_+$. Thus $||L \circ f^k \circ L(x)||^{1/k} ||f^k(L(z))||^{1/k} \ge n^{1/k}$. Taking the limit as $k \to \infty$ we see that $\rho(f) \ge \rho(L \circ f \circ L)^{-1}$. □

Proof of Theorem 2. Let $L : \operatorname{int}\mathbb{R}^n_+ \to \operatorname{int}\mathbb{R}^n_+$ be given by (2). The map $L \circ f \circ L$ is an order-preserving homogeneous of degree one map on $\operatorname{int}\mathbb{R}^n_+$. By [3, Theorem 2.10], $L \circ f \circ L$ extends continuously to an order-preserving map on the entire cone $\mathbb{R}^n_+$. By the Krein-Rutman theorem, the continuous extension of $L \circ f \circ L$ must have an eigenvector $y \in \mathbb{R}^n_+ \backslash \{0\}$ with eigenvalue $\tilde{\rho}^{-1}$ where $\tilde{\rho} = \rho(L \circ f \circ L)^{-1}$. By Lemma 3, $\tilde{\rho} \le \rho = \rho(f)$. Since L is a continuous order-reversing bijection from $\mathbb{R}^n_+$ onto $(0,\infty]^n$, it follows that f extends continuously to an order-preserving map on $(0,\infty]^n$. Let $z = L(y)$. Since $y \ne 0$, at least one entry of z is finite and $f(z) = L \circ L \circ f(L(y)) = L(L \circ f \circ L(y)) = L(\tilde{\rho}^{-1}(y)) = \tilde{\rho} z$. □

The continuous extension of an order-preserving homogeneous of degree one map to the set $(0,\infty]^n$ is the crucial insight in the proof above. This extension is noted in the context of order-preserving additively homogeneous maps in [1].

Example 1. Let $J_n(\lambda)$ denote the n-by-n Jordan matrix with eigenvalue $\lambda > 0$,

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

The linear transformation corresponding to $J_n(\lambda)$ maps the interior of $\mathbb{R}^n_+$ into itself. Up to scalar multiplication, the unique Perron eigenvector is $e_1 = [1, 0, \ldots, 0]^T$, and the spectral radius of $J_n(\lambda)$ is λ. The formal eigenvector corresponding to $J_n(\lambda)$ (which is unique up to scalar multiplication) is $[\infty, \ldots, \infty, 1]$. Thus, Theorem 2 tells us that $(J_n(\lambda)^k(x))_n \leq \lambda^k x_n$ for all $k \in \mathbb{N}$.

Example 2. The "DAD maps" were introduced in [8] to solve the problem of trying to find diagonal matrices D_1 and D_2 for a nonnegative m-by-n matrix A such that D_1AD_2 is doubly stochastic. Such matrices exist if and only if the nonlinear map $f : x \mapsto LA^TLAx$ has an eigenvector with all positive entries. This in turn occurs if and only if the matrix A is a direct sum of fully indecomposible matrices [2].

Note that the map f is order-preserving and homogenous of degree one and it is defined on the interior of $\mathbb{R}^n_+$. Therefore it must extend continuously to the boundary of $\mathbb{R}^n_+$, and this continuous extension can be computed using the convention that $(0)^{-1} = \infty$ and $(\infty)^{-1} = 0$.

Assume that A is an m-by-n nonnegative matrix such that every row and column contains at least one non-zero entry. By permuting the rows and columns of A, we may assume [5] that A has the following form,

$$A = \begin{bmatrix} A_1 & B_{12} & \ldots & B_{1\sigma} \\ & A_2 & \ldots & B_{2\sigma} \\ & 0 & \ddots & \vdots \\ & & & A_\sigma \end{bmatrix}$$

where each A_i is an m_i-by-n_i matrix with a corresponding DAD-map $f_i(x) = LA_i^TLA_ix$ with a unique eigenvector v_i (up to scaling) with all positive entries in $\mathbb{R}^{n_i}$ and eigenvalue $\lambda_i = n_i/m_i$. Furthermore $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\sigma$. If A is indecomposable (that is, there are no permutation matrices P and Q such that PAQ is a direct sum of two matrices), then the form above is unique. Note that the eigenvector v_σ can be extended to a formal eigenvector for the map $f(x) = LA^TLAx$ by letting

$$v = \begin{bmatrix} \infty \\ \vdots \\ \infty \\ v_\sigma \end{bmatrix}.$$

Then $f(v) = LA^TLAv = \lambda_\sigma v$. Note that the cone spectral radius of f is λ_1 [5] so the eigenvalue corresponding to the formal eigenvector may be strictly less than the cone spectral radius.

4. Weak Upper Bounds

Theorem 3. *Let $f : \operatorname{int} C \to \operatorname{int} C$ be order-preserving and homogeneous of degree one with $\rho(f) = \rho$. If C is a polyhedral cone and $x \in \operatorname{int} C$, then there is a $w \in C^* \backslash \{0\}$ such that $\langle f^k(x), w\rangle \leq \rho^k \langle x, w\rangle$ for all $k \in \mathbb{N}$.*

Proof. Fix $x_0 \in \operatorname{int} C$, and a linear functional $v \in \operatorname{int} C^*$. Let $f_\epsilon(x) = f(x) + \epsilon\langle x, v\rangle x_0$ for any $x \in \operatorname{int} C$. Let $\rho_\epsilon = \rho(f_\epsilon)$. By [6, Theorem 5.4.1], each f_ϵ has an eigenvector $x_\epsilon \in \operatorname{int} C$ with $f_\epsilon(x_\epsilon) = \rho_\epsilon x_\epsilon$. Since we are free to scale the eigenvectors x_ϵ, we will require that x_ϵ be the smallest scalar multiple of x_ϵ with the property that $x_\epsilon \geq x$. For each $\epsilon > 0$ there is a unit vector $w_\epsilon \in \operatorname{extr} C^*$ such that $\langle x_\epsilon, w_\epsilon\rangle = \langle x, w_\epsilon\rangle$. Since C is polyhedral, there are only finitely many extreme rays of C^*. We may choose a sequence ϵ_i such that $\epsilon_i \to 0$ and for all $i \in \mathbb{N}$, $w_{\epsilon_i} = w$ where w is a unit vector in $\operatorname{extr} C^*$. Then, $\langle x_{\epsilon_i}, w\rangle = \langle x, w\rangle$ for all i. We see that

$$\langle f^k(x), w\rangle \leq \langle f^k_{\epsilon_i}(x_{\epsilon_i}), w\rangle = \rho^k_{\epsilon_i}\langle x, w\rangle.$$

Since C is polyhedral, it follows that the cone spectral radius ρ_ϵ is continuous [6, Corollary 5.5.4]. This means that $\rho_\epsilon \to \rho$ as $\epsilon \to 0$. Therefore, $\langle f^k(x), w\rangle \leq \rho^k\langle x, w\rangle$ for all $k \in \mathbb{N}$. □

A stronger result than Theorem 3 can be obtained for linear maps with an invariant closed cone in $\mathbb{R}^n$ by using the adjoint.

Theorem 4. *Let C be a proper closed cone in $\mathbb{R}^n$. Let $T : C \to C$ be a linear map and let $T^* : C^* \to C^*$ denote the adjoint of T. Let $\rho = \rho(T)$ be the spectral radius of T. Then there is a $w \in C^* \backslash \{0\}$ such that $\langle T(x), w\rangle = \rho\langle x, w\rangle$ for all $x \in C$. In particular, w is a weak upper bound for the iterates of T.*

Proof. Note that the spectral radius of the adjoint $\rho(T^*) = \rho(T)$. Let w be an eigenvector of T^* with eigenvalue ρ, as is guaranteed to exist by the Krein-Rutman theorem. For any $x \in C$, $\langle T(x), w\rangle = \langle x, T^*(w)\rangle = \rho\langle x, w\rangle$. □

For an arbitrary proper closed cone $C \subset \mathbb{R}^n$, it is not yet known whether any order-preserving homogeneous of degree one map $f : \operatorname{int} C \to \operatorname{int} C$ must have a weak upper bound.

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Philip Chodrow, Swarthmore College
E-mail address: pchodro1@swarthmore.edu

Cole Franks, University of South Carolina
E-mail address: franksw@email.sc.edu

Brian Lins, Hampden-Sydney College
E-mail address: blins@hsc.edu